

Indecomposable modules of the intermediate series over $\mathcal{W}(a, b)^*$

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Abstract. For any complex parameters a, b , $\mathcal{W}(a, b)$ is the Lie algebra with basis $\{L_i, W_i \mid i \in \mathbb{Z}\}$ and relations $[L_i, L_j] = (j - i)L_{i+j}$, $[L_i, W_j] = (a + j + bi)W_{i+j}$, $[W_i, W_j] = 0$. In this paper, indecomposable modules of the intermediate series over $\mathcal{W}(a, b)$ are classified. It is also proved that an irreducible Harish-Chandra $\mathcal{W}(a, b)$ -module is either a highest/lowest weight module or a uniformly bounded module. Furthermore, if $a \notin \mathbb{Q}$, an irreducible weight $\mathcal{W}(a, b)$ -module is simply a *Vir*-module with trivial actions of W_k 's.

Key words: the algebra $\mathcal{W}(a, b)$, the Virasoro algebra, modules of the intermediate series

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1 Introduction

For any fixed complex numbers a, b , there exists a Lie algebra with basis $\{L_i, W_i \mid i \in \mathbb{Z}\}$ and Lie brackets

$$[L_i, L_j] = (j - i)L_{i+j}, \quad [L_i, W_j] = (a + j + bi)W_{i+j}, \quad [W_i, W_j] = 0. \quad (1.1)$$

This Lie algebra, known as the *algebra* $\mathcal{W}(a, b)$, is in fact the semi-direct product $\text{Vir} \ltimes A'_{a,b}$ of *Vir* by $A'_{a,b}$, where $\text{Vir} = \text{span}\{L_i \mid i \in \mathbb{Z}\}$ is the well-known centerless Virasoro algebra (or Witt algebra), and $A'_{a,b}$ is a module of the intermediate series defined in (2.2) (which is regarded as an abelian Lie algebra). Thus, from the definition, one immediately sees that this Lie algebra is closely related to the Virasoro algebra and its modules. Due to their extreme importance in mathematics and physics, representations of the Virasoro algebra (or higher rank Virasoro algebras, e.g., [10, 11]) have been widely studied in the mathematical and physical literatures. For instance, a classification of modules of the immediate series over the Virasoro algebra was given in [3], unitarizable modules and uniformly bounded modules with composition factors at most two were considered respectively in [1, 2], and a classification of Harish-Chandra modules over the Virasoro algebra was presented in [8] (see also, [1, 9]).

In order to investigate a classification of vertex operator algebras generated by weight 2 vectors, the *W*-algebra $W(2, 2)$, which is a special case of $\mathcal{W}(a, b)$ with $a = 0$, $b = -1$, was first introduced and studied in [14]. Later on, a classification of Harish-Chandra modules over $W(2, 2)$ was considered in [6]. Furthermore, the well-known twisted Heisenberg-Virasoro

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algebra (without some central elements), whose irreducible Harish-Chandra modules were classified in [7], is also a special case of $\mathcal{W}(a, b)$ with $a = b = 0$. Thus, $\mathcal{W}(a, b)$ generalizes many meaningful algebras, and it is very natural and desirable to consider representations of $\mathcal{W}(a, b)$.

Definition 1.1. A $\mathcal{W}(a, b)$ -module V is called

- a *weight module* if it admits a weight space decomposition $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$ (λ is called a *weight* of V in case $V_\lambda \neq 0$), where

$$V_\lambda = \{v_\lambda \mid L_0 v_\lambda = \lambda v_\lambda\} \quad \text{for } \lambda \in \mathbb{C}; \quad (1.2)$$

- a *highest* (resp., *lowest*) *weight module* with highest (resp., lowest) weight Λ if there exists $\Lambda \in \mathbb{C}$ such that V is generated by V_Λ , and $V_\lambda = 0$ for all $\lambda \in \mathbb{C}$ with $\lambda - \Lambda \in \mathbb{Z}_+ \setminus \{0\}$ (resp., $\Lambda - \lambda \in \mathbb{Z}_+ \setminus \{0\}$);
- a *Harish-Chandra module* if it is a weight module with $\dim V_\lambda < \infty$ for all $\lambda \in \mathbb{C}$;
- a *uniformly bounded module* if it is a Harish-Chandra module such that there exists some $N > 0$ with $\dim V_\lambda \leq N$ for all $\lambda \in \mathbb{C}$;
- a *module of the intermediate series* if V is a uniformly bounded module such that $\dim V_\lambda \leq 1$ for all $\lambda \in \mathbb{C}$.

The aim of the present paper is to give a classification of (not only irreducible but also) indecomposable modules of the intermediate series over $\mathcal{W}(a, b)$ and give some description of irreducible weight $\mathcal{W}(a, b)$ -modules. Our next goal is to give a classification of all irreducible Harish-Chandra $\mathcal{W}(a, b)$ -modules in some due time. The main techniques used in this paper are developed from that used in determining representations of higher rank Virasoro algebras [10, 11]. Similar techniques have been also used in determining representations of Block type Lie algebras and Schrödinger-Virasoro algebras [4, 13]. We would like to emphasize here that, as we shall see later on, due to the crucial fact that the parameter a is not necessarily an integer, modules of the intermediate series over $\mathcal{W}(a, b)$ may have a rather complicated structure, which is very different from that of the Virasoro algebra, whose indecomposable modules of the intermediate series are only slightly different than irreducible modules of the intermediate series.

Note that some special cases of $\mathcal{W}(a, b)$ naturally appear as subalgebras of many interesting infinite-dimensional graded Lie (super)algebras, e.g., the W -infinity algebra $\mathcal{W}_{1+\infty}$, some Block type Lie algebras, $N = 2$ super-Virasoro algebras, Schrödinger-Virasoro algebras [4, 5, 12, 13], etc. Just as results on representations of the Virasoro algebra are widely used in classifications of representations of Lie (super)algebras which contain the Virasoro algebra as a subalgebra, one can expect that results on representations of $\mathcal{W}(a, b)$ may be used in that of Lie (super)algebras which contain some $\mathcal{W}(a, b)$ as a subalgebra (this is also one of our motivations in presenting the results below). However, in order to be able to apply

our results to representations of Lie (super)algebras which contain $\mathcal{W}(a, b)$ as a subalgebra, it seems to be necessary to have a classification of (not only irreducible but also) indecomposable $\mathcal{W}(a, b)$ -modules of the intermediate series. This is why we consider indecomposable rather than irreducible modules here.

Our first main result in this paper is to give the following classification of indecomposable $\mathcal{W}(a, b)$ -modules of the intermediate series. Here and below, we always assume $b \neq 0$ since $\mathcal{W}(a, 0) \cong \mathcal{W}(a, 1)$ (cf. (3.1)) if $a \notin \mathbb{Z}$, and in case $a \in \mathbb{Z}$, $\mathcal{W}(a, 0) \cong \mathcal{W}(0, 0)$ is simply the twisted Heisenberg-Virasoro algebra, whose indecomposable modules of the intermediate series were considered in [7].

Theorem 1.2. *Let V be an indecomposable $\mathcal{W}(a, b)$ -module of the intermediate series. Then we have one of the following:*

- (1) *V is a Vir-module of the intermediate series (cf. (2.1)–(2.4)) with trivial actions of W_k 's.*
- (2) *$a \notin \mathbb{Z}$, $b \neq 0, 1$, and V is a sub-quotient of*

$$A(\lambda, \mu), B(\lambda, \mu), A_1(\gamma), A_2(\gamma), A_3(\gamma), B_1(\gamma), B_2(\gamma) \text{ or } B_3(\gamma), \quad (1.3)$$

defined in (3.2)–(3.9) for some $\lambda, \mu \in \mathbb{C}$, $\gamma \in \mathbb{C} \cup \{\infty\}$ (cf. Convention 2.1).

- (3) *$a \notin \mathbb{Z}$, $b = 1$, and V is a sub-quotient of a module in (1.3) or a sub-quotient of*

$$\tilde{A}(\lambda, \mu), \tilde{B}(\lambda, \mu), \tilde{A}_1(\gamma), \tilde{A}_2(\gamma), \tilde{A}_3(\gamma), \tilde{B}_1(\gamma), \tilde{B}_2(\gamma), \tilde{B}_3(\gamma) \text{ or } \overline{A}(\lambda, \mu), \quad (1.4)$$

defined in (3.10)–(3.17) and (6.4) for some $\lambda, \mu \in \mathbb{C}$, $\gamma \in \mathbb{C} \cup \{\infty\}$. Furthermore, $\overline{A}(\lambda, \mu)$ can occur only when $a \in \mathbb{Q}$.

- (4) *$a \in \mathbb{Z}$, $b = 1$, and V is a quotient of $\overline{A}(\lambda, \mu, c)$ for some $\lambda, \mu, c \in \mathbb{C}$, which has basis $\{v_m \mid m \in \mathbb{Z}\}$ with actions defined by*

$$\overline{A}(\lambda, \mu, c) : \quad L_k v_m = (\lambda + m + \mu k) v_{k+m}, \quad W_k v_m = \delta_{k+a, 0} c v_{m+k+a}. \quad (1.5)$$

The second main result is the following description of irreducible weight $\mathcal{W}(a, b)$ -modules.

Theorem 1.3. (1) *Any irreducible Harish-Chandra $\mathcal{W}(a, b)$ -module is either a uniformly bounded module or a highest/lowest weight module.*

- (2) *Let V be an irreducible $\mathcal{W}(a, b)$ -module of the intermediate series. Then we have one of the following:*
 - (i) *V is a Vir-module with trivial actions of W_k 's;*
 - (ii) *$a \in \mathbb{Q} \setminus \mathbb{Z}$, $b = 1$, and V is a quotient of $\overline{A}(\lambda, \mu)$ defined in (6.4) for some $\lambda, \mu \in \mathbb{C}$;*
 - (iii) *$a \in \mathbb{Z}$, and V is a quotient of $\overline{A}(\lambda, \mu, c)$ defined in (1.5) for some $\lambda, \mu, c \in \mathbb{C}$ with $c \neq 0$.*
- (3) *If $a \notin \mathbb{Q}$, then an irreducible weight $\mathcal{W}(a, b)$ -module (not necessarily a Harish-Chandra module) is simply a Vir-module with trivial action of W_k 's.*

The paper is arranged as follows. After presenting some notations, definitions and preliminary results in Section 2, we first list all possible maximal indecomposable $\mathcal{W}(a, b)$ -modules of the intermediate series with condition $a \notin \mathbb{Q}$ in Section 3. Then we give a proof of Theorem 1.2 in Sections 4 and 5 for the case $a \notin \mathbb{Q}$, and in Section 6 for the cases $a \in \mathbb{Q} \setminus \mathbb{Z}$ and $a \in \mathbb{Z}$ respectively. Finally, we give a proof of Theorem 1.3 in Section 7.

Throughout the paper, we denote by \mathbb{Z} , \mathbb{Z}_+ , \mathbb{Q} and \mathbb{C} the sets of integers, nonnegative integers, rational numbers and complex numbers respectively.

2 Preliminaries

From (1.1), one immediately sees that

$$Vir = \text{span}\{L_i \mid i \in \mathbb{Z}\}, \quad (2.1)$$

is the well-known centerless *Virasoro algebra*. An indecomposable Vir -module of the intermediate series [3] must be one of $A'_{\lambda, \mu}, A'(\gamma), B'(\gamma)$, $\lambda, \mu \in \mathbb{C}$, $\gamma \in \mathbb{C} \cup \{\infty\}$ (we add the prime to the notations in order to avoid the confusion with $\mathcal{W}(a, b)$ -modules to be introduced later), or one of their quotient submodules, where $A'_{\lambda, \mu}, A'(\gamma), B'(\gamma)$ all have a basis $\{v_n \mid n \in \mathbb{Z}\}$ such that

$$A'_{\lambda, \mu} : L_i v_j = (\lambda + j + \mu i) v_{i+j}, \quad (2.2)$$

$$A'(\gamma) : L_i v_0 = i(i + \gamma) v_i, \quad L_i v_j = (i + j) v_{i+j} \text{ for } j \neq 0, \quad (2.3)$$

$$B'(\gamma) : L_i v_{-i} = -i(i + \gamma) v_0, \quad L_i v_j = j v_{i+j} \text{ for } j \neq -i. \quad (2.4)$$

Here and below, for convenience, we use the following:

Convention 2.1. If $\gamma = \infty$, we always regard $i + \gamma$ as 1. Thus in fact $A'(\infty) = A'_{0,1}$ and $B'(\infty) = A'_{0,0}$.

Denote by $A''_{0,0}$ the unique nontrivial irreducible quotient of $A'_{0,0}$, and T the 1-dimensional trivial module.

Now let V be an indecomposable $\mathcal{W}(a, b)$ -module of the intermediate series, namely, the weight space V_λ defined by (1.2) satisfies $\dim V_\lambda \leq 1$ for all $\lambda \in \mathbb{C}$. Since $\{W_i \mid i \in \mathbb{Z}\}$ spans an Abelian ideal of $\mathcal{W}(a, b)$, if $W_i V = 0$ for all $i \in \mathbb{Z}$, then V is simply a Vir -module, whose structure is well-known. So, we shall always suppose

$$W_i V \neq 0 \text{ for some } i \in \mathbb{Z}. \quad (2.5)$$

We denote $P(V) = \{\mu \in \mathbb{C} \mid V_\mu \neq 0\}$, called the *set of weights of V* . Fix a weight $\lambda \in \mathbb{C}$, so that $V_\lambda \neq 0$. Using $[L_0, L_i] = iL_i$ and $[L_0, W_j] = (a + j)W_j$, it is easy to obtain that $L_j V_\lambda \in V_{\lambda+j}$ and $W_j V_\lambda \in V_{\lambda+a+j}$. Since V is indecomposable, for any two nonzero weight

vectors $u, v \in V$ with different weights, there exist weight vectors $v_0 := u, v_1, \dots, v_k := v$ and basis elements $x_1, \dots, x_k \in \{L_i, W_i \mid i \in \mathbb{Z}\}$ and nonzero numbers $a_1, \dots, a_k \in \mathbb{C}$, such that either $x_i v_{i-1} = a_i v_i$ or $a_i v_{i-1} = x_i v_i, i = 1, \dots, k$. This implies

$$P(V) \subset \{\lambda + ja + m \mid j, m \in \mathbb{Z}\}. \quad (2.6)$$

In case $a \notin \mathbb{Q}$, we have

$$V = \bigoplus_{-\infty < j < \infty} V^j, \quad V^j = \bigoplus_{m \in \mathbb{Z}} V_m^j, \quad V_m^j = \{v \in V \mid L_0 v = (\lambda + aj + m)v\}, \quad (2.7)$$

and $\dim V_m^j \leq 1$.

Let N_1 (resp., N_2) be the smallest (resp., largest) integer such that $V^{N_1+1}, V^{N_2-1} \neq 0$. Then $N_1 < N_2 - 1$, and $V = \bigoplus_{N_1 < j < N_2} V^j$, such that each V^j is a nonzero module of the intermediate series over Vir and $W_i V^j \subset V^{j+1}$ for $N_1 < j < N_2$. Note that N_1 (resp., N_2) can be $-\infty$ (resp., ∞). In case $N_1 < -\infty$, we can suppose $N_1 = -1$ if necessary. Define the *length* $\ell(V)$ of V to be $N_2 - N_1 + 1$.

In case $a \in \mathbb{Q} \setminus 0$, by shifting index of W_m if necessary, we can suppose $a = \frac{q}{p}$ with $1 \leq q < p$ and p, q are coprime. We can also suppose $V = \bigoplus_{j=0}^N V^j$ with $N < p$ (i.e., $N_1 = -1, N_2 = N + 1 \leq p$). We also denote $V^j = 0$ if $j < 0$ or $j > N$.

Let V be a $\mathcal{W}(a, b)$ -module of the intermediate series with decomposition (2.7). Take the subspace \mathbf{V} of V^* with decomposition $\mathbf{V} = \bigoplus_{-\infty < j < \infty} \mathbf{V}^j$ such that $\mathbf{V}^j = \bigoplus_{m \in \mathbb{Z}} \mathbf{V}_m^j$ and $\mathbf{V}_m^j = (V_m^j)^*$ (where “ $*$ ” stands for the “dual space”). Then \mathbf{V} is a $\mathcal{W}(a, b)$ -module of the intermediate series by defining for $x \in \mathcal{W}(a, b), \mathbf{v} \in \mathbf{V}$,

$$x\mathbf{v}(v) = -\mathbf{v}(xv), \quad \forall v \in V. \quad (2.8)$$

We simply call \mathbf{V} the *dual* $\mathcal{W}(a, b)$ -module of V .

In the following sections, we shall determine indecomposable $\mathcal{W}(a, b)$ -modules according to the cases $a \notin \mathbb{Q}, a \in \mathbb{Q} \setminus \mathbb{Z}$ and $a \in \mathbb{Z}$.

3 Modules of the intermediate series for case $a \notin \mathbb{Q}$

In this section, we consider the case $a \notin \mathbb{Q}$. In this case, we can assume $\ell(V) \geq 2$, otherwise V is simply a Vir -module. We can also suppose $b \neq 0$ because there exist an algebra isomorphism $\eta: \mathcal{W}(a, 0) \cong \mathcal{W}(a, 1)$ defined by

$$\eta(L_k^{(0)}) = L_k^{(1)}, \quad \eta(W_k^{(0)}) = (a + k)W_k^{(1)}, \quad \forall k \in \mathbb{Z}, \quad (3.1)$$

where the basis elements of $\mathcal{W}(a, i)$ are denoted by $L_k^{(i)}, W_k^{(i)}, i = 0, 1$.

We first list all possible maximal indecomposable $\mathcal{W}(a, b)$ -modules of the intermediate series which we obtained using complex computations (an indecomposable module is *maximal* if it cannot be strictly embedded in another indecomposable module). For any $\lambda, \mu \in \mathbb{C}$, it is easy to check that there exist indecomposable $\mathcal{W}(a, b)$ modules (which will be proved to be maximal indecomposable) $A(\lambda, \mu)$, $B(\lambda, \mu)$ of the intermediate series with bases $\{v_m^j \mid j, m \in \mathbb{Z}\}$, $\{v_m^j \mid m \in \mathbb{Z}, j = 0, 1\}$ respectively and actions:

$$A(\lambda, \mu) = \text{span}\{v_m^j \mid j, m \in \mathbb{Z}\} : \\ L_k v_m^j = (\lambda + aj + m + (\mu + jb)k)v_{k+m}^j, \quad W_k v_m^j = v_{k+m}^{j+1}, \quad \forall j, k, m \in \mathbb{Z}; \quad (3.2)$$

$$B(\lambda, \mu) = \text{span}\{v_m^j \mid m \in \mathbb{Z}, j = 0, 1\} : \\ L_k v_m^0 = (\lambda + m + \mu k)v_{k+m}^0, \quad W_k v_m^0 = (b(\lambda + m) - \mu(k + a))v_{m+k}^1, \\ L_k v_m^1 = (\lambda + a + m + (\mu + b + 1)k)v_{m+k}^1, \quad W_k v_m^1 = 0, \quad \forall k, m \in \mathbb{Z}. \quad (3.3)$$

Obviously, the dual module of $A(\lambda, \mu)$ is $A(-\lambda, 1 - \mu)$ and the dual module of $B(\lambda, \mu)$ is $B(-\lambda - a, -\mu - b)$, by taking the dual basis of $\{v_m^j \mid j, m \in \mathbb{Z}\}$ to be $\{(-1)^j w_{-m}^{-j} \mid j, m \in \mathbb{Z}\}$, i.e., $w_n^i(v_m^j) = (-1)^j \delta_{i, -j} \delta_{n, -m}$. Thus, no new modules can be produced by taking duals of $A(\lambda, \mu)$ and $B(\lambda, \mu)$.

For any $\gamma \in \mathbb{C} \cup \{\infty\}$, we have $\mathcal{W}(a, b)$ -modules $A_1(\gamma)$, $A_2(\gamma)$, $A_3(\gamma)$ with basis $\{v_m^j \mid j, m \in \mathbb{Z}\}$, $\{v_m^j \mid m \in \mathbb{Z}, j \geq -1\}$, $\{v_m^j \mid m \in \mathbb{Z}, j = -1, 0\}$ respectively and actions

$$A_1(\gamma) = \text{span}\{v_m^j \mid j, m \in \mathbb{Z}\} : \\ L_k v_m^0 = (m + k)v_{k+m}^0 \quad (m \neq 0), \quad L_k v_0^0 = k(k + \gamma)v_k^0, \\ L_k v_m^j = (aj + m + jbk)v_{k+m}^j \quad (j \neq 0), \\ W_k v_m^{-1} = (m + k)v_{m+k}^0, \quad W_k v_m^0 = \delta_{m,0}v_{k+m}^1, \quad W_k v_m^j = v_{k+m}^{j+1} \quad (j \neq 0, -1); \quad (3.4)$$

$$A_2(\gamma) = \text{span}\{v_m^j \mid j, m \in \mathbb{Z}, j \geq -1\} : \\ L_k v_m^{-1} = (-a + m - (1 + b)k)v_{m+k}^{-1}, \quad W_k v_m^{-1} = (m + k)(a + (1 + b)k + bm)v_{k+m}^0, \\ L_k v_m^0 = (m + k)v_{m+k}^0 \quad (m \neq 0), \quad L_k v_0^0 = k(k + \gamma)v_k^0, \quad W_k v_m^0 = \delta_{m,0}v_{m+k}^1, \\ L_k v_m^j = (ja + m + jbk)v_{m+k}^j \quad (j > 0), \quad W_k v_m^j = v_{m+k}^{j+1} \quad (j \neq -1, 0); \quad (3.5)$$

$$A_3(\gamma) = \text{span}\{v_m^j \mid m \in \mathbb{Z}, j = 0, 1\} : \\ L_k v_m^0 = (m + k)v_{m+k}^0 \quad (m \neq 0), \quad L_k v_0^0 = k(k + \gamma)v_k^0, \\ W_k v_m^0 = bv_{m+k}^1 \quad (m \neq 0), \quad W_k v_0^0 = (b\gamma - a - k)v_k^1, \\ L_k v_m^1 = (a + m + (b + 1)k)v_{m+k}^1, \quad W_k v_m^1 = 0. \quad (3.6)$$

The dual modules of $A_i(\gamma)$, $i = 1, 2, 3$ are the following.

$$B_1(\gamma) = \text{span}\{v_m^j \mid j, m \in \mathbb{Z}\} :$$

$$\begin{aligned}
L_k v_m^0 &= m v_{m+k}^0 \ (m \neq -k), \ L_k v_{-k}^0 = -k(k + \gamma) v_0^0, \\
L_k v_m^j &= (ja + m + (jb + 1)k) v_{m+k}^j \ (j \neq 0), \\
W_k v_m^{-1} &= \delta_{k+m,0} v_{k+m}^0, \ W_k v_m^0 = m v_{k+m}^1, \ W_k v_m^j = v_{k+m}^{j+1} \ (j \neq 0, -1);
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
B_2(\gamma) &= \text{span}\{v_m^j \mid j, m \in \mathbb{Z}, j \leq 1\} : \\
L_k v_m^0 &= m v_{m+k}^0 \ (m \neq -k), \ L_k v_{-k}^0 = -k(k + \gamma) v_0^0, \\
L_k v_m^j &= (ja + m + (jb + 1)k) v_{m+k}^j \ (j < 0), \\
L_k v_m^1 &= (a + m + (b + 2)k) v_{m+k}^1, \ W_k v_m^j = v_{k+m}^{j+1} \ (j < -1), \\
W_k v_m^{-1} &= \delta_{m+k,0} v_{k+m}^0, \ W_k v_m^0 = m(a + k - bm) v_{m+k}^1, \ W_k v_m^1 = 0;
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
B_3(\gamma) &= \text{span}\{v_m^j \mid m \in \mathbb{Z}, j = -1, 0\} : \\
L_k v_m^0 &= m v_{m+k}^0 \ (m \neq -k), \ L_k v_{-k}^0 = -k(k + \gamma) v_0^0, \ W_k v_m^0 = 0, \\
L_k v_m^{-1} &= (-a + m - bk) v_{m+k}^{-1}, \ W_k v_m^{-1} = v_{k+m}^0 \ (m \neq -k), \\
W_k v_{-k}^{-1} &= (b\gamma - a - k) v_0^0.
\end{aligned} \tag{3.9}$$

If $b \neq 0, 1$, the above are all maximal indecomposable $\mathcal{W}(a, b)$ -modules. However, when $b = 0$ or 1 , there will be more maximal indecomposable modules due to the fact that there exists the algebra isomorphism (3.1), and thus a $\mathcal{W}(a, 0)$ -module V becomes a $\mathcal{W}(a, 1)$ -module (denoted by \tilde{V} when there is no confusion), by the action $xv = \eta^{-1}(x)v$ for all $x \in \mathcal{W}(a, 1)$.

Now suppose $b = 1$. We already have maximal indecomposable $\mathcal{W}(a, 1)$ -modules $A(\lambda, \mu)$, $B(\lambda, \mu)$, $A_i(\gamma)$, $B_i(\gamma)$, $i = 1, 2, 3$. In addition, there will be maximal indecomposable $\mathcal{W}(a, 1)$ -modules derived from $\mathcal{W}(a, 0)$ -modules by using (3.1). First we have maximal indecomposable $\mathcal{W}(a, 1)$ -modules $\tilde{A}(\lambda, \mu)$, $\tilde{A}_1(\gamma)$, $\tilde{A}_3(\gamma)$, $\tilde{B}_1(\gamma)$, $\tilde{B}_3(\gamma)$ derived from $\mathcal{W}(a, 0)$ -modules $A(\lambda, \mu)$, $A_1(\gamma)$, $A_3(\gamma)$, $B_1(\gamma)$, $B_3(\gamma)$ respectively,

$$\begin{aligned}
\tilde{A}(\lambda, \mu) &= \text{span}\{v_m^j \mid j, m \in \mathbb{Z}\} : \\
L_k v_m^j &= (\lambda + aj + m + \mu k) v_{k+m}^j, \ W_k v_m^j = \frac{1}{a + k} v_{k+m}^{j+1};
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
\tilde{A}_1(\gamma) &= \text{span}\{v_m^j \mid j, m \in \mathbb{Z}\} : \\
L_k v_m^0 &= (m + k) v_{k+m}^0 \ (m \neq 0), \ L_k v_0^0 = k(k + \gamma) v_k^0, \\
L_k v_m^j &= (aj + m) v_{k+m}^j \ (j \neq 0), \ W_k v_m^j = \frac{1}{a + k} v_{k+m}^{j+1} \ (j \neq 0, -1), \\
W_k v_m^{-1} &= \frac{m + k}{a + k} v_{m+k}^0, \ W_k v_m^0 = \frac{\delta_{m,0}}{a + k} v_{k+m}^1;
\end{aligned} \tag{3.11}$$

$$\tilde{A}_3(\gamma) = \text{span}\{v_m^j \mid m \in \mathbb{Z}, j = 0, 1\} :$$

$$\begin{aligned}
L_k v_m^0 &= (m+k)v_{m+k}^0 \ (m \neq 0), \quad L_k v_0^0 = k(k+\gamma)v_k^0, \\
W_k v_m^0 &= 0 \ (m \neq 0), \quad W_k v_0^0 = v_k^1, \\
L_k v_m^1 &= (a+m+k)v_{m+k}^1, \quad W_k v_m^1 = 0;
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
\tilde{B}_1(\gamma) &= \text{span}\{v_m^j \mid m, j \in \mathbb{Z}\} : \\
L_k v_m^0 &= m v_{m+k}^0 \ (m \neq -k), \quad L_k v_{-k}^0 = -k(k+\gamma)v_0^0, \\
L_k v_m^j &= (ja+m+k)v_{m+k}^j \ (j \neq 0), \quad W_k v_m^{-1} = \frac{\delta_{m+k,0}}{a+k} v_{m+k}^0, \\
W_k v_m^0 &= \frac{m}{a+k} v_{m+k}^1, \quad W_k v_m^j = \frac{1}{a+k} v_{m+k}^{j+1} \ (j \neq 0, -1);
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
\tilde{B}_3(\gamma) &= \text{span}\{v_m^j \mid m \in \mathbb{Z}, j = -1, 0\} : \\
L_k v_m^0 &= m v_{m+k}^0 \ (m \neq -k), \quad L_k v_{-k}^0 = -k(k+\gamma)v_0^0, \quad W_k v_m^0 = 0, \\
L_k v_m^{-1} &= (-a+m)v_{m+k}^{-1}, \quad W_k v_m^{-1} = \frac{1}{a+k} v_{k+m}^0 \ (m \neq -k), \\
W_k v_{-k}^{-1} &= -v_0^0.
\end{aligned} \tag{3.14}$$

Note that in case $b = 0$, the indecomposable $\mathcal{W}(a, 0)$ -module $B(\lambda, \mu)$ of length 2 is not maximal, it is contained in a maximal indecomposable module (denoted by, say, M), but the maximal indecomposable $\mathcal{W}(a, 1)$ -module derived from M by using (3.1) is nothing but the $\mathcal{W}(a, 1)$ -module $A(\lambda, \mu)$. Thus we cannot produce any new maximal indecomposable $\mathcal{W}(a, 1)$ -module from the $\mathcal{W}(a, 0)$ -module $B(\lambda, \mu)$. However, we found a maximal indecomposable $\mathcal{W}(a, 0)$ -module of length 2, denoted by $\hat{B}(\lambda, \mu)$,

$$\begin{aligned}
\hat{B}(\lambda, \mu) &= \text{span}\{v_m^j \mid m \in \mathbb{Z}, j = 0, 1\} : \\
L_k v_m^0 &= (\lambda+m)v_{k+m}^0, \quad W_k v_m^0 = ((am - \lambda k)\mu + a + k)v_{m+k}^1, \\
L_k v_m^1 &= (\lambda + a + m + k)v_{k+m}^1, \quad W_k v_m^1 = 0.
\end{aligned}$$

Furthermore, the indecomposable $\mathcal{W}(a, 0)$ -modules $A_2(\gamma)$, $B_2(\gamma)$ are not maximal, they are contained respectively in the maximal indecomposable $\mathcal{W}(a, 0)$ -modules $\hat{A}_2(\gamma)$, $\hat{B}_2(\gamma)$,

$$\begin{aligned}
\hat{A}_2(\gamma) &= \text{span}\{v_m^j \mid j, m \in \mathbb{Z}\} : \\
L_k v_m^0 &= (m+k)v_{k+m}^0 \ (m \neq 0), \quad L_k v_0^0 = k(k+\gamma)v_k^0, \quad W_k v_m^0 = \delta_{m,0}v_{m+k}^1, \\
L_k v_m^j &= (ja+m)v_{m+k}^j \ (j > 0), \quad L_k v_m^j = (ja+m+jk)v_{m+k}^j \ (j < 0), \\
W_k v_m^{-1} &= (m+k)(a+k)v_{m+k}^0, \quad W_k v_m^j = v_{m+k}^{j+1} \ (j > 0), \\
W_k v_m^j &= (a+k)v_{m+k}^{j+1} \ (j < -1);
\end{aligned}$$

$$\hat{B}_2(\gamma) = \text{span}\{v_m^j \mid j, m \in \mathbb{Z}\} :$$

$$\begin{aligned}
L_k v_m^0 &= m v_{m+k}^0 \ (m \neq -k), \ L_k v_{-k}^0 = -k(k+\gamma) v_0^0, \ W_k v_m^0 = m(a+k) v_{m+k}^1, \\
L_k v_m^j &= (ja+m+k) v_{m+k}^j \ (j < 0), \ L_k v_m^j = (ja+m+(j+1)k) v_{m+k}^j \ (j > 0), \\
W_k v_m^{-1} &= \delta_{m+k,0} v_{m+k}^0, \ W_k v_m^j = v_{m+k}^{j+1} \ (j < -1), \ W_k v_m^j = (a+k) v_{m+k}^{j+1} \ (j > 1).
\end{aligned}$$

Thus, we have $\mathcal{W}(a, 1)$ -modules by using (3.1), denoted by $\tilde{B}(\lambda, \mu)$, $\tilde{A}_2(\gamma)$, $\tilde{B}_2(\gamma)$,

$$\tilde{B}(\lambda, \mu) = \text{span}\{v_m^j \mid m \in \mathbb{Z}, j = 0, 1\} :$$

$$\begin{aligned}
L_k v_m^0 &= (\lambda+m) v_{k+m}^0, \ W_k v_m^0 = \frac{(am-\lambda k)\mu+a+k}{a+k} v_{m+k}^1, \\
L_k v_m^1 &= (\lambda+a+m+k) v_{k+m}^1, \ W_k v_m^1 = 0;
\end{aligned} \tag{3.15}$$

$$\tilde{A}_2(\gamma) = \text{span}\{v_m^j \mid j, m \in \mathbb{Z}\} :$$

$$\begin{aligned}
L_k v_m^0 &= (m+k) v_{k+m}^0 \ (m \neq 0), \ L_k v_0^0 = k(k+\gamma) v_k^0, \ W_k v_m^0 = \frac{\delta_{m,0}}{a+k} v_{m+k}^1, \\
L_k v_m^j &= (ja+m) v_{m+k}^j \ (j > 0), \ L_k v_m^j = (ja+m+jk) v_{m+k}^j \ (j < 0), \\
W_k v_m^{-1} &= (m+k) v_{m+k}^0, \ W_k v_m^j = \frac{1}{a+k} v_{m+k}^{j+1} \ (j > 0), \ W_k v_m^j = v_{m+k}^{j+1} \ (j < -1);
\end{aligned} \tag{3.16}$$

$$\tilde{B}_2(\gamma) = \text{span}\{v_m^j \mid j, m \in \mathbb{Z}\} :$$

$$\begin{aligned}
L_k v_m^0 &= m v_{m+k}^0 \ (m \neq -k), \ L_k v_{-k}^0 = -k(k+\gamma) v_0^0, \ W_k v_m^0 = m v_{m+k}^1, \\
L_k v_m^j &= (ja+m+k) v_{m+k}^j \ (j < 0), \ L_k v_m^j = (ja+m+(j+1)k) v_{m+k}^j \ (j > 0), \\
W_k v_m^{-1} &= \frac{\delta_{m+k,0}}{a+k} v_{m+k}^0, \ W_k v_m^j = \frac{1}{a+k} v_{m+k}^{j+1} \ (j < -1), \ W_k v_m^j = v_{m+k}^{j+1} \ (j > 1).
\end{aligned} \tag{3.17}$$

The following two sections are devoted to the proof of Theorem 1.2 in case $a \notin \mathbb{Q}$. Note that for any $j \in \mathbb{Z}$ with $N_1 < j < N_2$, the Vir -module V^j can be in general any of $A'_{\lambda, \mu}$, $A'(\gamma)$, $B'(\gamma)$, $A''_{0,0}$, T , $A''_{0,0} \oplus T$. We split the possible V^j into two cases:

- (1) each V^j is an irreducible Vir -module of type $A'_{\lambda, \mu}$.
- (2) At least one V^j is not an irreducible Vir -module of type $A'_{\lambda, \mu}$.

4 Proof of Theorem 1.2 in the first case for $a \notin \mathbb{Q}$

In this section, we always suppose that $a \notin \mathbb{Q}$ and each V^j for $N_1 < j < N_2$ is an irreducible Vir -module of the intermediate series of type $A'_{\lambda, \mu}$ (later on, we shall consider all possible deformations). Thus we can choose a basis $\{v_m^j \mid m \in \mathbb{Z}\}$ of V^j such that

$$L_k v_m^j = \ell_{k,m}^j v_{m+k}^j, \ \ell_{k,m}^j = \lambda + aj + m + k\mu_j, \ \text{and} \ W_k v_m^j = \mathbf{w}_{k,m}^j v_{m+k}^{j+1}, \tag{4.1}$$

for some $\mu_j, \mathbf{w}_{k,m}^j \in \mathbb{C}$ (cf. (2.7)).

- Remark 4.1.** (1) We use the bold symbol \mathbf{w} to emphasis that $\mathbf{w}_{k,m}^j$'s are unknown variables to be determined.
- (2) We have used the convention that if an undefined symbol technically appears in an expression, we always treat it as zero; for instance, $v_m^j = 0$ if $j \leq N_1$ or $j \geq N_2$.
- (3) By assumption, we have

$$\lambda + aj \notin \mathbb{Z} \text{ or } \mu_j \notin \{0, 1\} \text{ for } N_1 < j < N_2. \quad (4.2)$$

Fix $j \in \mathbb{Z}$ with $N_1 < j < N_2 - 1$. Since V is indecomposable, there exist some i_j, m_j such that

$$\mathbf{w}_{i_j, m_j}^j \neq 0. \quad (4.3)$$

Applying $(a + k + bi)W_{i+k} = [L_i, W_k]$ to v_m^j , we obtain

$$(a + k + bi)\mathbf{w}_{i+k, m}^j = \ell_{i, k+m}^{j+1} \mathbf{w}_{k, m}^j - \mathbf{w}_{k, m+i}^j \ell_{i, m}^j. \quad (4.4)$$

Applying $(a + k + b(i_1 + i_2))[L_{i_1}, [L_{i_2}, W_k]] = (a + i_2 + k + bi_1)(a + k + bi_2)[L_{i_1+i_2}, W_k]$ to v_m^j , we obtain

$$\begin{aligned} (a + k + b(i_1 + i_2)) & \left(\ell_{i_1, i_2+k+m}^{j+1} (\ell_{i_2, k+m}^{j+1} \mathbf{w}_{k, m}^j - \mathbf{w}_{k, i_2+m}^j \ell_{i_2, m}^j) \right. \\ & \quad \left. - (\ell_{i_2, i_1+k+m}^{j+1} \mathbf{w}_{k, i_1+m}^j - \mathbf{w}_{k, i_1+i_2+m}^j \ell_{i_2, i_1+m}^j) \ell_{i_1, m}^j \right) \\ & = (a + i_2 + k + bi_1)(a + k + bi_2) (\ell_{i_1+i_2, k+m}^{j+1} \mathbf{w}_{k, m}^j - \mathbf{w}_{k, i_1+i_2+m}^j \ell_{i_1+i_2, m}^j). \end{aligned} \quad (4.5)$$

Lemma 4.2. Let $j \in \mathbb{Z}$ with $N_1 < j < N_2 - 1$. For any $k \in \mathbb{Z}$, there exist infinitely many values of m such that $\mathbf{w}_{k, m}^j \neq 0$.

Proof. Suppose conversely there exist some $k_0, N \in \mathbb{Z}$ such that $\mathbf{w}_{k_0, m}^j = 0$ for all m with $|m| > N$. Let n be any integer. Take any i_1, i_2 and $k = k_0, m = n - i_1$ such that

$$|n - i_1|, |n + i_2|, |n - i_1 + i_2| > N. \quad (4.6)$$

Then all terms in (4.5) vanish except the term containing $\mathbf{w}_{k_0, n}^j$, and we obtain

$$(a + k_0 + b(i_1 + i_2))(\lambda + a(j + 1) + k_0 + n + i_2\mu_{j+1})(\lambda + aj + n - i_1 + i_1\mu_j)\mathbf{w}_{k_0, n}^j = 0. \quad (4.7)$$

Using condition (4.2) and the fact that $a \notin \mathbb{Q}$, we can always choose i_1, i_2 satisfying (4.6) such that the coefficient of $\mathbf{w}_{k_0, n}^j$ in (4.7) is not zero. Thus $\mathbf{w}_{k_0, n}^j = 0$. Take $i = i_0$ and $k = k_0$ in (4.4), we obtain $\mathbf{w}_{i_0+k_0, m}^j = 0$ if $a + k_0 + bi_0 \neq 0$. Assume $a + k_0 + bi_0 = 0$. Then $b \neq 1$ since $a \notin \mathbb{Q}$. We choose $i = i_0 + j, k = k_0 - j$ for any $j \notin \{0, -i_0\}$, we again obtain $\mathbf{w}_{i_0+k_0, m}^j = 0$. This proves $\mathbf{w}_{k, m}^j = 0$ for any k, m , a contradiction with (4.3). \square

Remark 4.3. In order to simplify notation in the following discussions, without loss of generality, we can always suppose $j = 0$ by shifting indices j if necessary. When we state the results, we shall take the general case into account.

Lemma 4.4. *We have the following possibilities (cf. Remark 4.5):*

- (1) $\mu_{j+1} = \mu_j + b + 1$, and $\mathbf{w}_{k,m}^j = b(\lambda + ja + m) - (a + k)\mu_j$,
- (2) $b \neq 1$, $\mu_{j+1} = \mu_j + b$, and $\mathbf{w}_{k,m}^j = 1$,
- (3) $b = 1$, $\mu_j \neq 0$, $\mu_{j+1} = \mu_j + 1$, and $\mathbf{w}_{k,m}^j = 1$,
- (4) $b = 1$, $\mu_j = 0$, $\mu_{j+1} = 1$, and $\mathbf{w}_{k,m}^j = \frac{cam - c(\lambda + ja)k + a + k}{a + k}$ for some $c \in \mathbb{C}$,
- (5) $b = 1$, $\mu_{j+1} = \mu_j \neq 0, 1$, and $\mathbf{w}_{k,m}^j = \frac{1}{a + k}$,
- (6) $b = 1$, $\mu_{j+1} = \mu_j = 0$, and $\mathbf{w}_{k,m}^j = \frac{cam - c(\lambda + ja)k + a + k}{(a + k)(\lambda + ja + a + m + k)}$ for some $c \in \mathbb{C}$,
- (7) $b = 1$, $\mu_{j+1} = \mu_j = 1$, and $\mathbf{w}_{k,m}^j = \frac{cam - c(\lambda + ja)k + a + k}{(a + k)(\lambda + ja + m)}$ for some $c \in \mathbb{C}$,
- (8) $b = 1$, $\mu_j = 1$, $\mu_{j+1} = 0$, and $\mathbf{w}_{k,m}^j = \frac{cam - c(\lambda + ja)k + a + k}{(\lambda + ja + m)(a + k)(\lambda + ja + a + m + k)}$ for some $c \in \mathbb{C}$.

Proof. By Remark 4.3, we can suppose $j = 0$. Taking $k = 0$ and taking the data (i_1, i_2, m) in (4.5) to be $(i, -i, n)$, $(i, i, n - i)$, $(-i, -i, n + i)$ respectively, we obtain a system of three linear equations on $\mathbf{w}_{0,n-i}^0$, $\mathbf{w}_{0,n}^0$, $\mathbf{w}_{0,n+i}^0$,

$$f_i^{s,-1} \mathbf{w}_{0,n-i}^0 + f_i^{s,0} \mathbf{w}_{0,n}^0 + f_i^{s,1} \mathbf{w}_{0,n+i}^0 = 0, \quad s = 1, 2, 3, \quad \text{where,} \quad (4.8)$$

$$\begin{aligned} f_i^{1,-1} &= a(i\mu_0 - \lambda - n)(i(\mu_1 - 1) + a + \lambda + n), \\ f_i^{1,0} &= a((b(b-1) - \mu_0(\mu_0 - 1) - \mu_1(\mu_1 - 1))i^2 + 2(\lambda + n)(a + \lambda + n)), \\ f_i^{1,1} &= f_{-i}^{1,-1}, \\ f_i^{2,-1} &= b(1 + b - 2\mu_1(2 + b + \mu_1))i^3 + (a(1 - b(b+1) + \mu_1(\mu_1 - 3)) - b(\lambda + n)(3 + b - 4\mu_1))i^2 \\ &\quad - (a^2 - 2a(\lambda + n)(b - 1 + \mu_1) - 2b(\lambda + n)^2)i + a(\lambda + n)(a + \lambda + n), \\ f_i^{2,0} &= 2(2bi + a)(i(\mu_0 - 1) - \lambda - n)(i\mu_1 + \lambda + n + a), \\ f_i^{2,1} &= b(-1 - b + 2\mu_0(b + \mu_0))i^3 + (a(-1 + \mu_0(\mu_0 + 1) + 2b(-1 + 2\mu_0)) + b(\lambda + n)(b - 1 + 4\mu_0))i^2 \\ &\quad - (a^2 + 2a(\lambda + n)(1 - b - \mu_1) - 2b(\lambda + n)^2)i + a(\lambda + n)(a + \lambda + n), \\ f_i^{3,-1} &= f_{-i}^{2,1}, \quad f_i^{3,0} = f_{-i}^{3,0}, \quad f_i^{3,1} = f_{-i}^{2,-1}. \end{aligned} \quad (4.9)$$

Denote the determinant of the coefficients by $\Delta(i)$, which is zero for all i by Lemma 4.2,

$$\begin{aligned} 0 &= \Delta(i) = -ai^6(\mu_1 - \mu_0 - b)(\mu_1 - \mu_0 - b - 1)(\Delta_2 i^2 + \Delta_1(\lambda + n) + \Delta_0), \quad \text{where,} \quad (4.10) \\ \Delta_2 &= 4b^2(-1 + \mu_0 + \mu_1)(-b\mu_0 + b^2\mu_0 + \mu_0^2 - \mu_0^3 + 2\mu_1 - b\mu_1 - b^2\mu_1 + 2b\mu_0\mu_1 - \mu_0^2\mu_1 - 3\mu_1^2 + \mu_0\mu_1^2 + \mu_1^3), \\ \Delta_1 &= 2ab(b - 1)(-2 + b + b^2 + 5\mu_0 - 2b\mu_0 - 3\mu_0^2 + 7\mu_1 + 2b\mu_1 - 6\mu_0\mu_1 - 3\mu_1^2), \\ \Delta_0 &= a^2(2b - 3b^2 + b^4 + 2\mu_0 - 10b\mu_0 + 10b^2\mu_0 - 2b^3\mu_0 - 3\mu_0^2 + 10b\mu_0^2 - 6b^2\mu_0^2 - 2b\mu_0^3 + \mu_0^4 + 6\mu_1 \end{aligned}$$

$$\begin{aligned}
& -10b\mu_1 + 2b^2\mu_1 + 2b^3\mu_1 - 10\mu_0\mu_1 + 18b\mu_0\mu_1 - 6b^2\mu_0\mu_1 + 2\mu_0^2\mu_1 - 6b\mu_0^2\mu_1 + 2\mu_0^3\mu_1 \\
& - 11\mu_1^2 + 8b\mu_1^2 + 8\mu_0\mu_1^2 - 6b\mu_0\mu_1^2 + 6\mu_1^3 - 2b\mu_1^3 - 2\mu_0\mu_1^3 - \mu_1^4).
\end{aligned}$$

Note that for any given n_0 , if we replace the basis element v_m^j by $v_m'^j = v_{m+n_0}^j$, then this amounts to replacing λ by $\lambda + n_0$. Thus, (4.10) implies

$$\mu_1 = \mu_0 + b, \text{ or } \mu_1 = \mu_0 + b + 1, \text{ or } \Delta_2 = \Delta_1 = \Delta_0 = 0. \quad (4.11)$$

The possible solutions of (4.11) are

$$\begin{aligned}
& \{\mu_1 = \mu_0 + b\}, \{\mu_1 = \mu_0 + b + 1\}, \{b = 1, \mu_1 = \mu_0\}, \{b = 1, \mu_1 = 1 - \mu_0\}, \\
& \{b = 1, \mu_0 = \frac{9-t^2}{8}, \mu_1 = \frac{(t+1)(t+3)}{8}\}, \{\mu_0 = \frac{(b+1)(b+2)}{2(2b+1)}, \mu_1 = \frac{b(1-b)}{2(2b+1)}\}, \quad (4.12)
\end{aligned}$$

and

$$\{\mu_0 = 0, \mu_1 = b + 2\}, \{\mu_0 = 1, \mu_1 = b\}, \{\mu_0 = 1 - b, \mu_1 = 0\}, \{\mu_0 = -1 - b, \mu_1 = 1\}. \quad (4.13)$$

Note that when $\mu_j = 0$ or 1 , we can always change basis elements v_m^j 's such that μ_j becomes 1 or 0 . Thus all four cases in (4.13) can be regarded as special cases of the first two cases of (4.12).

Now for all possible cases in (4.12), we need to process the following.

- (i) Using Lemma 4.2, by shifting the index m of basis elements v_m^0 's if necessary, we can suppose $\mathbf{w}_{0,0}^0 \neq 0$. Then using (4.8), we can solve for $\mathbf{w}_{0,m}^0$ (in terms of $\mathbf{w}_{0,0}^0$).
- (ii) Then we use (4.4) to solve $\mathbf{w}_{k,m}^0$.
- (iii) Finally we verify that the solution $\mathbf{w}_{k,m}^0$ satisfies (4.4).

Now we consider six cases in (4.12) case by case.

Case 1. $\mu_1 = \mu_0 + b$.

Assume $\mathbf{w}_{0,0}^0 = 1$ by rescaling basis elements v_m^0 's. First we suppose $b \neq 0, 1$. By (4.8), we obtain for all $m \in \mathbb{Z}$, $\mathbf{w}_{0,m}^0 = 1$. Taking $k = 0$ in (4.4), we have

$$\begin{aligned}
(a + bi)\mathbf{w}_{i,m}^0 &= (\lambda + m + a + \mu_1 i)\mathbf{w}_{0,m}^0 - (\lambda + m + \mu_0 i)\mathbf{w}_{0,m+i}^0 \\
&= \lambda + m + a + (b + \mu_0)i - (\lambda + m + \mu_0 i) = a + bi.
\end{aligned} \quad (4.14)$$

For any $i_0 \in \mathbb{Z}$, if $a + bi_0 \neq 0$, then $\mathbf{w}_{i_0,m}^0 = 1$ for all $m \in \mathbb{Z}$ by (4.14). Suppose $a + bi_0 = 0$. If $i_0 \neq 1$, then $\mathbf{w}_{i,m}^0 = 1$ for all $i \neq i_0$. Taking $k = 1, i = i_0 - 1$ in (4.4),

$$(1 - b)\mathbf{w}_{i_0,m}^0 = (\lambda + m + 1 + a + (i_0 - 1)\mu_1)\mathbf{w}_{1,m}^0 - (\lambda + m + (i_0 - 1)\mu_0)\mathbf{w}_{1,m+i_0-1}^0$$

$$\begin{aligned}
&= \lambda + m + 1 + a + (i_0 - 1)(b + \mu_0) - (\lambda + m + (i_0 - 1)\mu_0) \\
&= 1 + a + (i_0 - 1)b = 1 - b.
\end{aligned}$$

Thus, $\mathbf{w}_{i_0, m}^0 = 1$. If $i_0 = 1$, by taking $i = 2, k = -1$ in (4.4), we again obtain $\mathbf{w}_{1, m}^0 = 1$. Thus Lemma 4.4(2) holds.

Now we suppose $b = 1$. If $\mu_0 \neq 0$, by (4.8), we easily get $\mathbf{w}_{0, 0}^0 - \mathbf{w}_{0, i}^0 = 0$, i.e., $\mathbf{w}_{0, m}^0 = 1$ for all $m \in \mathbb{Z}$. Then by taking $k = 0$ in (4.4), we get

$$\begin{aligned}
(a + i)\mathbf{w}_{i, m}^0 &= (\lambda + m + a + \mu_1 i)\mathbf{w}_{0, m}^0 - (\lambda + m + \mu_0 i)\mathbf{w}_{0, m+i}^0 \\
&= \lambda + m + a + (1 + \mu_0)i - (\lambda + m + \mu_0 i) = a + i,
\end{aligned}$$

i.e., $\mathbf{w}_{k, m}^0 = 1$ for all $k, m \in \mathbb{Z}$. Thus Lemma 4.4(3) holds.

Hence we assume $\mu_0 = 0$. Then (4.8) gives $\mathbf{w}_{0, m-i}^0 - 2\mathbf{w}_{0, m}^0 + \mathbf{w}_{0, m+i}^0 = 0$. In particular, $\mathbf{w}_{0, m+1}^0 - \mathbf{w}_{0, m}^0$ is a constant. Thus, $\mathbf{w}_{0, m}^0 = cm + 1$ for all $m \in \mathbb{Z}$ and for some $c \in \mathbb{C}$. Taking $k = 0$ in (4.4), we have

$$\begin{aligned}
(a + i)\mathbf{w}_{i, m}^0 &= (\lambda + m + a + \mu_1 i)\mathbf{w}_{0, m}^0 - (\lambda + m + \mu_0 i)\mathbf{w}_{0, m+i}^0 \\
&= (\lambda + m + a + i)(cm + 1) - (\lambda + m)(cm + ci + 1) \\
&= cam - c\lambda i + a + i.
\end{aligned}$$

So, $\mathbf{w}_{k, m}^0 = \frac{cam - c\lambda k + a + k}{a + k}$. It is easy to check that the solution satisfies (4.4) and Lemma 4.4(4) holds.

Case 2. $\mu_1 = b + \mu_0 + 1$.

Taking $n = 0$ in (4.8), we have

$$(b(i + \lambda) - a\mu_0)\mathbf{w}_{0, 0}^0 - (b\lambda - a\mu_0)\mathbf{w}_{0, i}^0 = 0.$$

Since $\mathbf{w}_{0, 0}^0 \neq 0$ by the assumption, we must have $b\lambda - a\mu_0 \neq 0$. We can assume $\mathbf{w}_{0, m}^0 = b(m + \lambda) - a\mu_0$ by rescaling basis elements v_m 's. Taking $k = 0$ in (4.4), we have

$$\begin{aligned}
(a + bi)\mathbf{w}_{i, m}^0 &= (\lambda + m + a + \mu_1 i)\mathbf{w}_{0, m}^0 - (\lambda + m + \mu_0 i)\mathbf{w}_{0, m+i}^0 \\
&= (\lambda + m + a + (b + 1 + \mu_0)i)(b(m + \lambda) - a\mu_0) \\
&\quad - (\lambda + m + \mu_0 i)(b(m + i + \lambda) - a\mu_0) \\
&= (a + bi)(b(m + \lambda) - (i + a)\mu_0).
\end{aligned} \tag{4.15}$$

Thus Lemma 4.4(1) holds for all $k \in \mathbb{Z}$ with $a + bk \neq 0$.

Assume $k_0 \in \mathbb{Z}$ such that $a + bk_0 \neq 0$. We first assume $k_0 \neq 1$. Then $b \neq 1$ since $a \notin \mathbb{Z}$. Using the fact that $a = -bk_0$ and taking $i = k_0 - 1, k = 1$ in (4.4), we have

$$(1 - b)\mathbf{w}_{k_0, m}^0 = (\lambda + m + 1 + a + (k_0 - 1)\mu_1)\mathbf{w}_{1, m}^0 - (\lambda + m + (k_0 - 1)\mu_0)\mathbf{w}_{1, m+k_0-1}^0$$

$$\begin{aligned}
&= (\lambda + m + 1 + a + (k_0 - 1)(b + 1 + \mu_0))(b(m + \lambda) - (1 + a)\mu_0) \\
&\quad - (\lambda + m + (k_0 - 1)\mu_0)(b(m + k_0 - 1 + \lambda) - (1 + a)\mu_0) \\
&= (a + bk_0 - b + 1)(b(m + \lambda) - \mu_0(k_0 + a)) \\
&= (1 - b)(b(m + \lambda) - \mu_0(k_0 + a)).
\end{aligned}$$

Thus Lemma 4.4(1) also holds for k_0 . Now assume $k_0 = 1$. Taking $i = k_0 + 1, k = -1$ in (4.4), we again conclude that $\mathbf{w}_{k_0,m}^0 = b(m + \lambda) - \mu_0(k_0 + a)$, and so Lemma 4.4(1) holds.

Case 3. $b = 1, \mu_1 = \mu_0$.

First assume $\mu_0 \neq 0, 1$. By (4.8), we have $\mathbf{w}_{0,m}^0 = \mathbf{w}_{0,0}^0 = \frac{1}{a}$ by rescaling basis elements v_m^0 's. Taking $k = 0$ in (4.4), we have

$$\begin{aligned}
(a + i)\mathbf{w}_{i,m}^0 &= (\lambda + m + a + \mu_1 i)\mathbf{w}_{0,m}^0 - (\lambda + m + \mu_0 i)\mathbf{w}_{0,m+i}^0 \\
&= \frac{\lambda + m + a + \mu_0 i - (\lambda + m + \mu_0 i)}{a} = 1.
\end{aligned}$$

Thus Lemma 4.4(5) holds.

Now assume $\mu_0 = \mu_1 = 0$. If we replace the basis elements v_m^1 's of V^1 by $\tilde{v}_m^1 = \frac{1}{\lambda+a+m}v_m^1$, then μ_1 becomes 1, which becomes a special case considered in Case 1 with $b = 1$. Thus from Lemma 4.4(4), we obtain Lemma 4.4(6) by using the fact that $\tilde{v}_m^1 = \frac{1}{\lambda+a+m}v_m^1$. Similarly, we have Lemma 4.4(7) if $\mu_0 = \mu_1 = 1$.

Case 4. $b = 1, \mu_1 = 1 - \mu_0$.

First assume $\mu_0 \neq 0, 1$, and suppose $\mathbf{w}_{0,0}^0 = 1$. Taking $n = 0$ in (4.8), we easily get $(\mu_0 - 1)\mu_0(\mathbf{w}_{0,0}^0 - \mathbf{w}_{0,i}^0) = 0$. Thus $\mathbf{w}_{0,k}^0 = 1$ for all $k \in \mathbb{Z}$. Taking $k = 0$ in (4.4), we have

$$\begin{aligned}
(a + i)\mathbf{w}_{i,m}^0 &= (\lambda + m + a + \mu_1 i)\mathbf{w}_{0,m}^0 - (\lambda + m + \mu_0 i)\mathbf{w}_{0,m+i}^0 \\
&= a + i - 2\mu_0 i.
\end{aligned}$$

Thus, $\mathbf{w}_{k,m}^0 = \frac{a+k-2\mu_0 k}{a+k}$. Using this in (4.4), we obtain $\mu_0(2\mu_0 - 1) = 0$, i.e., $\mu_0 = \frac{1}{2}$, which is a special case in Case 3.

The case $\mu_0 = 0$ is a special case of Case 2. It remains to consider the case $\mu_0 = 1$. In this case, if we replace the basis elements v_m^0 's of V^0 by $\tilde{v}_m^0 = (\lambda + m)v_m^0$, then μ_0 becomes to 0, thus we obtain Lemma 4.4(8) by Lemma 4.4(6).

Case 5. $b = 1, \mu_0 = \frac{9-t^2}{8}, \mu_1 = \frac{(t+1)(t+3)}{8}$.

By (4.8), we have

$$g^1 \mathbf{w}_{0,n}^0 - g^2 \mathbf{w}_{0,n+i}^0 = 0, \quad \text{where,} \tag{4.16}$$

$$\begin{aligned}
g^1 &= (t-1)(1+t)(3+t)(16a^2(t-3) - i^2(1+t)^2(t(2+t) - 11) + 4ai(t-1)(t(2+t) - 7) \\
&\quad + (n+\lambda)(32a(t-1) - 16i(t(2+t) - 7)) - 64(n+\lambda)^2); \\
g^2 &= (t-1)(1+t)(3+t)(16a^2(t-3) + i^2(t-3)(5+t)(t(2+t) - 11) \\
&\quad + 32a(t-1)(n+\lambda) - 64(n+\lambda)^2).
\end{aligned}$$

Taking the data (n, i) in (4.16) to be $(n+i, -i)$, we obtain another equation involving $\mathbf{w}_{0,n}^0$ and $\mathbf{w}_{0,n+i}^0$. By Lemma 4.2, the determinant of the coefficients is equal to zero, i.e.,

$$16(a-i)i^2(a+i)(t-3)(t-1)^2(1+t)^2(3+t)^2(5+t)(t^2+2t-7)(t^2+2t-11)=0.$$

Thus, $t = -5, -1, -3, 1, 3, -1 \pm 2\sqrt{2}, -1 \pm 2\sqrt{3}$. All these are special cases of Cases 1–3.

Case 6. $\mu_0 = \frac{(b+1)(b+2)}{2(2b+1)}, \mu_1 = \frac{b(1-b)}{2(2b+1)}.$

Similar to Case 5, we obtain that the determinant of the coefficients is zero, i.e.,

$$\begin{aligned}
0 &= \frac{3i^6 a^2 (a-bi)(a-i-bi)(a+bi)(a+i+bi)}{4(1+2b)^6} \times \\
&\quad (b-1)^2 b^2 (b+1)^2 (b+2)(3b+2)(3b^2+2b+1)(3b^2+4b+2).
\end{aligned}$$

Thus, $b = -2, -1, 0, 1, -\frac{2}{3}$, or $3b^2+2b+1=0$ or $3b^2+4b+2=0$. All these except the case $b = -\frac{2}{3}$ are special cases of Cases 1–3. Hence, assume $b = -\frac{2}{3}$. By (4.8),

$$(a-i-n-\lambda)(2a+i+n+\lambda)(a+2(i+n+\lambda))\mathbf{w}_{0,n}^0 - (a-n-\lambda)(2a+n+\lambda)(a+2(n+\lambda))\mathbf{w}_{0,n+i}^0 = 0.$$

After solving $\mathbf{w}_{0,n}^0$, and taking $k = 0$ in (4.4), we obtain the solution for $\mathbf{w}_{k,m}^0$, which contradicts (4.4). This completes the proof of the Lemma 4.4. \square

Remark 4.5. Under the condition (4.2), Lemma 4.4(6)–(8) can be regarded as Lemma 4.4(4) by re-choosing bases of V^0, V^1 as we have seen in the above proof. However, we need to state them separately because we shall need to consider all possible deformations (i.e., dropping condition (4.2)) later.

If $\ell(V) = 2$, by Remark 4.5 and by considering cases in Lemma 4.4(1)–(5), we obtain that V is a sub-quotient of $A(\lambda, \mu), B(\lambda, \mu), \tilde{A}(\lambda, \mu), \tilde{B}(\lambda, \mu)$. Hence we have Theorem 1.2. Thus in the rest of this section, we assume $\ell(V) \geq 3$.

Lemma 4.6. (1) *If $b \neq 1$, then V is a sub-quotient of a module of the form $A(\lambda, \mu)$.*

(2) *If $b = 1$, then V is a sub-quotient of a module of the form $A(\lambda, \mu)$ or $\tilde{A}(\lambda, \mu)$.*

Proof. Applying $[W_{k_1}, W_{k_2}] = 0$ to v_m^j for $N_1 < j < N_2 - 2$, we obtain

$$\mathbf{w}_{k_1, k_2+m}^{j+1} \mathbf{w}_{k_2, m}^j = \mathbf{w}_{k_2, k_1+m}^{j+1} \mathbf{w}_{k_1, m}^j. \quad (4.17)$$

Case 1. $b \neq 0, 1$.

By Lemma 4.4, there are only two possibilities for μ_j and μ_{j+1} , i.e, $\mu_{j+1} = \mu_j + b + 1$ or $\mu_{j+1} = \mu_j + b$. We want to prove $\mu_{j+1} = \mu_j + b$ for all j . If $\mu_{j+1} = \mu_j + b + 1$ and $\mu_{j+2} = b + \mu_{j+1} + 1$ for some j , then (4.17) gives

$$\begin{aligned} & (b(\lambda + (j+1)a + k_2 + m) - (a + k_1)\mu_{j+1})(b(\lambda + ja + m) - (a + k_2)\mu_j) \\ &= (b(\lambda + (j+1)a + k_1 + m) - (a + k_2)\mu_{j+1})(b(\lambda + ja + m) - (a + k_1)\mu_j). \end{aligned}$$

From this, we obtain $b = 0$, a contradiction with the assumption. If $\mu_{j+1} = b + \mu_j + 1$ and $\mu_{j+2} = b + \mu_{j+1}$, then (4.17) gives

$$b(\lambda + ja + m) - (a + k_2)\mu_j = b(\lambda + ja + m) - (a + k_1)\mu_j,$$

which implies $\mu_j = 0$. By changing the basis elements v_m^j 's, we can take μ_j to be 1, thus $\mu_{j+1} = b + \mu_j$. If $\mu_{j+1} = b + \mu_j$ and $\mu_{j+2} = b + \mu_{j+1} + 1$, then (4.17) gives

$$b(\lambda + (j+1)a + k_2 + m) - (a + k_1)\mu_{j+1} = b(\lambda + (j+1)a + k_1 + m) - (a + k_2)\mu_{j+1},$$

which gives $\mu_{j+1} = -b$. Thus, $\mu_j = -2b$ and $\mu_{j+2} = 1$. By changing the basis elements v_m^{j+2} 's, we can take μ_{j+2} to be 0, thus $\mu_{j+2} = b + \mu_{j+1}$. Therefore, Lemma 4.6(1) holds.

Case 2. $b = 1$.

By Lemma 4.4, for each pair $(j, j+1)$, there are 7 possibilities, thus in principle, there are 49 possibilities for the pairs $(j, j+1)$, $(j+1, j+2)$. However, by taking (4.17) into account, we have only the following 19 possibilities.

1. If $\mu_{j+1} = \mu_j + 2$, $\mu_{j+2} = 1 + \mu_{j+1}$, then $\mu_j = 0$, $\mathbf{w}_{k,m}^j = \lambda + ja + m$, $\mathbf{w}_{k,m}^{j+1} = 1$.
2. If $\mu_j = -1$, $\mu_{j+1} = 1$, $\mu_{j+2} = 1$, then $\mathbf{w}_{k,m}^j = \lambda + (j+1)a + m + k$, $\mathbf{w}_{k,m}^{j+1} = \frac{1}{\lambda + (j+1)a + m}$.
3. If $\mu_j = -1$, $\mu_{j+1} = 1$, $\mu_{j+2} = 0$, then $\mathbf{w}_{k,m}^j = \lambda + (j+1)a + m + k$, $\mathbf{w}_{k,m}^{j+1} = \frac{1}{(\lambda + (j+1)a + m)(\lambda + ja + 2a + m + k)}$.
4. If $\mu_j = -2$, $\mu_{j+1} = -1$, $\mu_{j+2} = 1$, then $\mathbf{w}_{k,m}^j = 1$, $\mathbf{w}_{k,m}^{j+1} = \lambda + (j+1)a + m + k$.
5. If $\mu_{j+1} = \mu_j + 1$, $\mu_{j+2} = 1 + \mu_{j+1}$, then $\mathbf{w}_{k,m}^j = \mathbf{w}_{k,m}^{j+1} = 1$.
6. If $\mu_j = -1$, $\mu_{j+1} = 0$, $\mu_{j+2} = 1$, then $\mathbf{w}_{k,m}^j = \mathbf{w}_{k,m}^{j+1} = 1$.
7. If $\mu_j = -1$, $\mu_{j+1} = 0$, $\mu_{j+2} = 0$, then $\mathbf{w}_{k,m}^j = 1$, $\mathbf{w}_{k,m}^{j+1} = \frac{1}{\lambda + (j+2)a + m + k}$.
8. If $\mu_j = 0$, $\mu_{j+1} = 1$, $\mu_{j+2} = 2$, then $\mathbf{w}_{k,m}^j = \mathbf{w}_{k,m}^{j+1} = 1$.
9. If $\mu_j = 0$, $\mu_{j+1} = 1$, $\mu_{j+2} = 1$, then $\mathbf{w}_{k,m}^j = \frac{\lambda + ja + m}{a + k}$, $\mathbf{w}_{k,m}^{j+1} = \frac{1}{a + k}$.
10. If $\mu_j = 0$, $\mu_{j+1} = 1$, $\mu_{j+2} = 0$, then $\mathbf{w}_{k,m}^j = \frac{\lambda + (j+1)a + m + k}{a + k}$, $\mathbf{w}_{k,m}^{j+1} = \frac{1}{(a + k)(\lambda + (j+1)a + m)}$.

11. If $\mu_j = \mu_{j+1} = \mu_{j+2}$, then $w_{k,m}^j = w_{k,m}^{j+1} = \frac{1}{a+k}$.
12. If $\mu_j = 0, \mu_{j+1} = 0, \mu_{j+2} = 2$, then $w_{k,m}^j = \frac{1}{\lambda+(j+1)a+m+k}, w_{k,m}^{j+1} = \lambda + (j+1)a + m$.
13. If $\mu_j = \mu_{j+1} = 0, \mu_{j+2} = 1$, then $w_{k,m}^j = \frac{1}{a+k}, w_{k,m}^{j+1} = \frac{\lambda+(j+2)a+m+k}{a+k}$.
14. If $\mu_j = \mu_{j+1} = \mu_{j+2} = 0$, then $w_{k,m}^j = w_{k,m}^{j+1} = \frac{1}{a+k}$.
15. If $\mu_j = \mu_{j+1} = 1, \mu_{j+2} = 2$, then $w_{k,m}^j = \frac{1}{\lambda+ja+m}, w_{k,m}^{j+1} = 1$.
16. If $\mu_j = \mu_{j+1} = \mu_{j+2} = 1$, then $w_{k,m}^j = w_{k,m}^{j+1} = \frac{1}{a+k}$.
17. If $\mu_j = 1, \mu_{j+1} = 0, \mu_{j+2} = 2$, then $w_{k,m}^j = \frac{1}{(\lambda+ja+m)(\lambda+ja+a+m+k)}, w_{k,m}^{j+1} = \lambda + (j+1)a + m$.
18. If $\mu_j = 1, \mu_{j+1} = 0, \mu_{j+2} = 1$, then $w_{k,m}^j = \frac{1}{(\lambda+(j+1)a+m+k)(a+k)}, w_{k,m}^{j+1} = \frac{\lambda+(j+1)a+m}{a+k}$.
19. If $\mu_j = 1, \mu_{j+1} = 0, \mu_{j+2} = 0$, then $w_{k,m}^j = \frac{1}{(\lambda+ja+m)(a+k)}, w_{k,m}^{j+1} = \frac{1}{a+k}$.

In fact, all those possibilities are equivalent to the following two possibilities by changing the basis elements v_m^{j+1}, v_m^{j+2} if necessary.

- (1) $\mu_{j+2} = \mu_{j+1} + 1 = \mu_j + 2$, and $w_{k,m}^j = w_{k,m}^{j+1} = 1$.
- (2) $b = 1, \mu_{j+2} = \mu_{j+1} = \mu_j$, and $w_{k,m}^j = w_{k,m}^{j+1} = \frac{1}{a+k}$.

By induction on j , we complete the proof. \square

Now we have determined all possible structures of V under the assumption in this section.

5 Proof of the Theorem 1.2 in the second case for $a \notin \mathbb{Q}$

Now we consider the case that $a \notin \mathbb{Q}$ and there exists some j_0 such that the Vir -module V^{j_0} is either reducible or a composition factor of $A'_{0,0}$. Then all weights of V^{j_0} are integers. For any $j \neq j_0$ with $N_1 < j < N_2$, since the weights of V^j are in $a(j - j_0) + \mathbb{Z}$, we see that V^j is an irreducible Vir -module of type $A'_{\lambda,\mu}$. Therefore, such j_0 is unique. By shifting indices if necessary, we can suppose $j_0 = 0$. Thus for any j with $N_1 < j < N_2 - 1$ and $j, j+1 \neq 0$, (4.1) holds with $\lambda = 0$. From this, we obtain:

Remark 5.1. All results in Section 4 hold for j with $N_1 < j < N_2 - 1$ and $j, j+1 \neq 0$.

We need to consider the following five possible cases:

- (a) $V^0 = T = \mathbb{C}v_0^0$ is the 1-dimensional trivial Vir -module,
- (b) $V^0 = A''_{0,0}$ is the nontrivial composition factor of Vir -module $A'_{0,0}$,
- (c) $V^0 = A''_{0,0} \oplus T$,
- (d) $V^0 = B'(\gamma)$ or $A'_{0,0}$ (note that $A'_{0,0}$ can be regarded as the case $B(\gamma)$ by interpreting $i + \gamma$ as 1 when $\gamma = \infty$ in (2.3)),
- (e) $V^0 = A'(\gamma)$ or $A'_{0,1}$.

5.1 The case $V^0 = T$

Lemma 5.2. *If $V^1 \neq 0$ (i.e., $N_2 > 1$), then $V^{-1} = 0$ (i.e., $N_1 = -1$).*

Proof. Denote $v_m^1 = W_m v_0^0$. Then

$$L_i v_m^1 = [L_i, W_m] v_0^0 = (a + m + bi) v_{i+m}^1. \quad (5.1)$$

Thus $\mu_1 = b$. If $V^{-1} \neq 0$, we suppose $W_i v_m^{-1} = \delta_{i,-m} w_i v_0^0$ for some $w_i \in \mathbb{C}$. Then for $i \neq k$, we have $0 = [W_k, W_i] v_{-i}^{-1} = w_i W_k v_0^0 = w_i v_k^1$, i.e., $w_i = 0$, a contradiction with the indecomposable condition on V (cf. (4.3)). Thus $V^{-1} = 0$. \square

Lemma 5.3. *If $V^1 \neq 0$, then $\mu_j = jb$ for any $0 < j < N_2$ and $\mathbf{w}_{i,m}^j = 1$ if $j < N_2 - 1$. Thus V is a sub-quotient module of $A(0, 0)$ (cf. (3.2)).*

Proof. We already have $\mu_1 = b$. So assume $N_2 > 2$. From $0 = [W_i, W_k] v_0^0 = (\mathbf{w}_{i,k}^1 - \mathbf{w}_{k,i}^1) v_{i+k}^2$, we obtain $\mathbf{w}_{i,k}^1 = \mathbf{w}_{k,i}^1$. By Lemma 4.4 (cf. Remark 5.1), we obtain $\mu_2 = \mu_1 + b$ and $\mathbf{w}_{i,m}^1 = 1$. Now the result can be proved by induction on j . \square

Dually, if $V^{-1} \neq 0$ (then $V^1 = 0$), V is a sub-quotient module of $A(0, 1)$.

5.2 The case $V^0 = A''_{0,0}$

First assume $V^1 \neq 0$. In this case, we can suppose (4.1) also holds for $j = 0$ with $\mu_0 = 0$, $v_0^0 = 0$ and $\mathbf{w}_{k,0}^0 = 0$, $k \in \mathbb{Z}$. We claim

$$W_0 v_m^0 = \mathbf{w}_{0,m}^0 v_m^1 \neq 0 \text{ for some } m \text{ with } m \neq 0. \quad (5.2)$$

If not, applying $[L_i, W_0] = (a + bi)W_i$ to v_m^0 , we obtain $(a + bi)\mathbf{w}_{i,m}^0 = 0$, i.e., $\mathbf{w}_{i,m}^0 = 0$ for $i \neq -\frac{a}{b}$. If $i_0 = -\frac{a}{b} \in \mathbb{Z}$, then $b \neq 1$ since $a \notin \mathbb{Z}$. Applying $[L_i, W_{i_0-i}] = (a + i_0 - i + bi)W_{i_0}$ to v_m^0 , we have $(1 - b)(i_0 - i)\mathbf{w}_{i_0,m}^0 = 0$ for $i \in \mathbb{Z}$. Thus $\mathbf{w}_{i,m}^0 = 0$ for all $i, m \in \mathbb{Z}$, a contradiction with (4.3). Hence, (5.2) holds.

Lemma 5.4. *If $V^1 \neq 0$, we have the following possibilities:*

- (1) $\mu_1 = b = 1$, and $\mathbf{w}_{k,m}^0 = \frac{m}{a+k}$,
- (2) $\mu_1 = b + 1$, and $\mathbf{w}_{k,m}^0 = m$,
- (3) $\mu_1 = b + 2$, and $\mathbf{w}_{k,m}^0 = m(a + k - bm)$.

Proof. As in the proof of Lemma 4.4, we have (4.12) or (4.13) with $\mu_0 = 0$. All these cases can be regarded as follows,

$$(b, \mu_0, \mu_1) = (b, 0, b), \quad (b, 0, b + 1), \quad (b, 0, b + 2). \quad (5.3)$$

First consider the case $(b, \mu_0, \mu_1) = (b, 0, b)$. Assume $b \neq 0, 1$. By (4.8), we have $\mathbf{w}_{0,m}^0 = \mathbf{w}_{0,m+i}^0$ for $m \neq 0, \pm i$. Thus by rescaling basis elements of V^0 if necessary, we can assume $\mathbf{w}_{0,m}^0 = 1$ for $m \neq 0$. Applying $[L_{-i}, [L_i, W_0]] = (a + i - bi)(a + bi)W_0$ to v_i^0 , we obtain

$$(2(a + i) + i(\mu_1 - b)(1 - \mu_1 - b))\mathbf{w}_{0,i}^0 = (a + 2i - \mu_1 i)\mathbf{w}_{0,2i}^0, \quad (5.4)$$

which is a contraction. This proves $b = 1$. By (4.8) or as in Case 1 in the proof of Lemma 4.4, we have $\mathbf{w}_{0,n-i}^0 - 2\mathbf{w}_{0,n}^0 + \mathbf{w}_{0,n+i}^0 = 0$ for $n \neq 0, \pm i$. Thus, $\mathbf{w}_{0,n}^0 = cn + c'$ for $n \neq 0$ and some $c, c' \in \mathbb{C}$. By (5.4), $c' = 0$. Applying $[L_i, W_0] = (a + i)W_i$ to v_m^0 , we have $\mathbf{w}_{k,m}^0 = \frac{am}{a+k}$. By rescaling the basis elements v_m^0 's, we have $\mathbf{w}_{k,m}^0 = \frac{m}{a+k}$, i.e., Lemma 5.4(1) holds.

Now consider the case $(b, \mu_0, \mu_1) = (b, 0, b+1)$. By (4.8), we have $(m+i)\mathbf{w}_{0,m}^0 = m\mathbf{w}_{0,m+i}^0$ for $m \neq 0, \pm i$. Thus, we can assume $\mathbf{w}_{0,m}^0 = m$. Similar to the above, we obtain $\mathbf{w}_{k,m}^0 = m$, i.e., Lemma 5.4(1) holds.

Finally consider the case $(b, \mu_0, \mu_1) = (b, 0, b+2)$. By (4.8), we have $(m+i)(a-b(i+m))\mathbf{w}_{0,m}^0 = m(a-bm)\mathbf{w}_{0,m+i}^0$ for $m \neq 0, \pm i$. Thus, we can assume $\mathbf{w}_{0,m}^0 = m(a-bm)$. Similar to the above, we have $\mathbf{w}_{k,m}^0 = m(a+k-bm)$, i.e., Lemma 5.4(3) holds. \square

Now suppose $V^{-1} \neq 0$. In this case, (4.1) also holds for $j = -1$ with $\mu_0 = 0$, $v_0^0 = 0$ and $\mathbf{w}_{k,-k}^{-1} = 0$, $k \in \mathbb{Z}$.

Lemma 5.5. *If $V^{-1} \neq 0$, we have the following possibilities:*

- (1) $\mu_{-1} = -b$, and $\mathbf{w}_{k,m}^{-1} = 1 - \delta_{m+k,0}$,
- (2) $\mu_{-1} = -1 - b$, and $\mathbf{w}_{k,m}^{-1} = (1 - \delta_{m+k,0})(a + (1+b)k + bm)$,
- (3) $b = 1$, $\mu_{-1} = 0$, and $\mathbf{w}_{k,m}^{-1} = \frac{1 - \delta_{m+k,0}}{a+k}$.

Proof. By (4.12) or (4.13) with $\mu_0 = 0$, we have

$$(b, \mu_{-1}, \mu_0) = (b, -b, 0), (b, -1 - b, 0), (b, 1 - b, 0). \quad (5.5)$$

First assume $(b, \mu_{-1}, \mu_0) = (b, -b, 0)$. By (4.8), we have $\mathbf{w}_{0,m}^{-1} = \mathbf{w}_{0,m+i}^{-1}$ for $m \neq 0, \pm i$. Thus, we can assume $\mathbf{w}_{0,m}^{-1} = 1$ for $m \neq 0$. Applying $[L_i, W_0] = (a + bi)W_i$ to v_m^{-1} gives $\mathbf{w}_{k,m}^{-1} = 1 - \delta_{m+k,0}$, i.e., Lemma 5.5(1) holds.

Next assume $(b, \mu_{-1}, \mu_0) = (b, -1 - b, 0)$. By (4.8), we have $(a + b(i + m))\mathbf{w}_{0,m}^{-1} = (a + bm)\mathbf{w}_{0,m+i}^{-1}$ for $m \neq 0, \pm i$. Thus we can assume $\mathbf{w}_{0,m}^{-1} = a + bm$ for $m \neq 0$, and so $\mathbf{w}_{k,m}^{-1} = (1 - \delta_{m+k,0})(a + (1+b)k + bm)$, i.e., Lemma 5.5(2) holds.

Now assume $(b, \mu_{-1}, \mu_0) = (b, 1 - b, 0)$. If $b \neq 0, 1$, then by (4.8), we have $(m+i)\mathbf{w}_{0,m}^{-1} = m\mathbf{w}_{0,m+i}^{-1}$ for $m \neq 0, \pm i$, i.e., $\mathbf{w}_{0,m}^{-1} = \frac{1}{m}$ for $m \neq 0$ by rescaling the basis elements. However,

by applying $[L_{-i}, [L_i, W_0] = (a + bi)(a + i - bi)W_0$ to v_i^{-1} , we obtain

$$(2(i - a) + i(b - \mu_{-1})(b + \mu_{-1} - 1))\mathbf{w}_{0,i}^{-1} = 2(-a + i + \mu_{-1}i)\mathbf{w}_{0,2i}^{-1},$$

which is a contradiction. Thus $b = 1$. Then (4.8) gives $(m - i)\mathbf{w}_{0,m-i}^{-1} - 2m\mathbf{w}_{0,m}^{-1} + (m + i)\mathbf{w}_{0,m+i}^{-1} = 0$ for $m \neq 0, \pm i$. Thus $\mathbf{w}_{0,m}^{-1} = \frac{cm + c'}{m}$ for $m \neq 0$ and some $c, c' \in \mathbb{C}$. Then (4.8) gives $\mathbf{w}_{k,m}^{-1} = \frac{ac}{a+k} - \frac{c'}{m+k}$ for $m + k \neq 0$. Applying $[L_i, W_j] = (a + j + bi)W_{i+j}$ to v_{-j}^{-1} , we have $c' = 0$. Thus, $\mathbf{w}_{k,m}^{-1} = \frac{1 - \delta_{m+k,0}}{a+k}$ by rescaling the basis elements, i.e., Lemma 5.5(3) holds. \square

Lemma 5.6. (1) If $V^1 \neq 0$, then $V^{-1} = 0$.

(2) V is a sub-quotient module of $B(0, 0)$, $B(-a, -b - 1)$, $\tilde{B}(0, \mu)$, $A(0, 0)$, $\tilde{A}(0, 0)$, $B_i(\gamma)$, $\tilde{B}_i(\gamma)$.

Proof. In order to obtain Lemma 5.6(1), we consider (4.17) with $j = -1$, and use Lemmas 5.4, 5.5. Then the result can be obtained as in the proof of Lemma 4.6.

Now Considering all the possibilities in Lemma 4.4 with $j = 1$, and using (4.17) with $j = 0$, we obtain Lemma 5.6(2) as in the proof of Lemma 4.6. \square

5.3 The case $V^0 = A''_{0,0} \oplus T$

As in the proofs of Lemmas 5.2, 5.6, we have $V^1 = 0$ or $V^{-1} = 0$.

Lemma 5.7. (1) If $V^{-1} = 0$, $V^1 \neq 0$, then $\mu_1 = b$.

(2) There does not exist an indecomposable $\mathcal{W}(a, b)$ -module V with $V^{-1} = 0$, $V^1 \neq 0$.

Proof. We claim

$$\mathbf{w}_{m_0,0}^0 \neq 0 \text{ for some } m_0 \in \mathbb{Z}. \quad (5.6)$$

Otherwise, V would be a decomposable $\mathcal{W}(a, b)$ -module. Applying $[L_i, W_0] - (a + bi)W_i = 0$, $[L_{-i}, W_i] - (a + i - bi)W_0 = 0$ to v_0^0 , we obtain

$$(a + bi)\mathbf{w}_{i,0}^0 - (a + \mu_1 i)\mathbf{w}_{0,0}^0 = 0, \quad (a + i - \mu_1 i)\mathbf{w}_{i,0}^0 - (a + i - bi)\mathbf{w}_{0,0}^0 = 0.$$

The determinant, denoted by Δ' , of coefficients of the above linear equations must be zero, i.e., $\Delta' = i^2(\mu_1 - b)(1 - \mu_1 - b) = 0$. Thus, $\mu_1 = b$ or $\mu_1 = 1 - b$. By (4.4) with $j = m = 0$, we obtain $\mu_1 = b$ or $b = 1, \mu_1 = 0$. Hence, by changing the basis elements v_m^1 's if necessary, we can always suppose $\mu_1 = b$. This proves Lemma 5.7(1).

As in the proof of Lemma 5.4, we have $b = 1$, and $\mathbf{w}_{k,m}^0 = \frac{am}{a+k}$ for $m \neq 0$. Then by (4.4), we would have $\mathbf{w}_{k,0}^0 = 0$ for all k , a contradiction with (5.6). This proves Lemma 5.7(2). \square

Dually, there does exist an indecomposable $\mathcal{W}(a, b)$ -module V with $V^1 = 0$, $V^{-1} \neq 0$.

5.4 The case $V^0 = B'(\gamma)$ or $A'_{0,0}$

By Convention 2.1, we can assume $V^0 = B'(\gamma)$.

Lemma 5.8. *If $V^1 \neq 0$, then we have the results in Lemma 5.4.*

Proof. We claim

$$\mathbf{w}_{0,m_0}^0 \neq 0 \text{ for some } 0 \neq m_0 \in \mathbb{Z}. \quad (5.7)$$

Otherwise, we may assume $\mathbf{w}_{0,0}^0 = 1$ by (4.3). As in (5.3), we only need to consider three cases $\mu_1 = b, b+1, b+2$. First assume $\mu_1 = b \neq 1$, applying $[L_k, W_0] = (a+bk)W_k$ to v_m^0, v_0^0, v_{-k}^0 respectively, we have $\mathbf{w}_{k,m}^0 = 0, \mathbf{w}_{k,0}^0 = 1, (a+bk)\mathbf{w}_{k,-k}^0 = -(a-k+bk)$ for $m, k \neq 0$ with $m \neq -k$. Using this results and applying $[L_i, W_j] = (a+j+bi)W_{i+j}$ to v_{-i-j}^0 , we would obtain a contradiction. Similarly, we would obtain a contradiction for other cases. Thus we have (5.7). Then similar to the proof of Lemma 5.4, we obtain the lemma. \square

Lemma 5.9. *If $V^{-1} \neq 0$, then $\mu_{-1} = 1-b$ and $\mathbf{w}_{k,m}^{-1} = \delta_{k+m,0}$.*

Proof. As in (5.5), we need to consider three cases $\mu_{-1} = -b, -1-b, 1-b$. First assume $\mathbf{w}_{0,m_0}^{-1} \neq 0$ for some $0 \neq m_0 \in \mathbb{Z}$. Similar to the proof of Lemma 5.5, we can solve $\mathbf{w}_{0,m}^{-1}$ for $m \neq 0$. Then applying $[L_k, W_0] = (a+bk)W_k$ to $v_m^{-1}, v_0^{-1}, v_{-k}^{-1}$ respectively with $m \neq -k, 0$, we can solve $\mathbf{w}_{k,m}^{-1}$. Then using (4.4) with $m \neq -i-j, -i, -j, 0$ and $m = -i-j, -i, -j, 0$ respectively, we would obtain a contradiction for all the cases in Lemma 5.5. Thus $\mathbf{w}_{0,m}^{-1} = 0$ for $m \neq 0$, and we can assume $\mathbf{w}_{0,0}^{-1} = 1$ by (4.3). Then as the proof above, we have the lemma. \square

Now similar to the proofs of Lemmas 5.2, 5.6, we obtain that V is a sub-quotient module of $B(0,0), \tilde{B}(0,\mu), B_i(\gamma), \tilde{B}_i(\gamma), i = 1, 2, 3$.

5.5 The case $V^0 = A'(\gamma)$ or $A'_{0,1}$

Dually to the previous, we obtain that V is a sub-quotient module of $B(0,1), \tilde{B}(-a,\mu), A_i(\gamma), \tilde{A}_i(\gamma), i = 1, 2, 3$.

Now we have determined all possible structures of V under the assumption in this section.

6 Modules of the intermediate series for case $a \in \mathbb{Q}$

First we suppose $a \notin \mathbb{Z}$. Then we can write

$$a = \frac{q}{p} \text{ with } p, q \in \mathbb{Z} \setminus \{0\}, p \geq 2 \text{ and } p, q \text{ are coprime}, \quad (6.1)$$

and we can assume $b \neq 0$ since $\mathcal{W}(a, 0) \cong \mathcal{W}(a, 1)$. We can suppose

$$V = \bigoplus_{j=N_0}^{N_0+N} V^j, \quad \text{and } V^j = 0 \text{ if } j < N_0 \text{ or } j > N_0 + N, \quad (6.2)$$

for some $N_0, N \in \mathbb{Z}$ with $1 \leq N \leq p-1$. Note that for any fix $N_0, N \in \mathbb{Z}$ with $1 \leq N \leq p-1$, all modules defined in (3.2)–(3.17) for the case $a \notin \mathbb{Q}$ remain to be $\mathcal{W}(a, b)$ -modules for the case $a \in \mathbb{Q}$ under the additional conditions:

$$N_0 \leq j \leq N_0 + N, \quad \text{and } W_k v_m^{N_0+N} = 0 \text{ for all } k, m \in \mathbb{Z}. \quad (6.3)$$

We use the same symbols to denote these modules. In addition, we have another type of modules, denoted by $\overline{A}(\lambda, \mu)$, with basis $\{v_m^j \mid j, m \in \mathbb{Z}, 0 \leq j \leq p-1\}$ and actions:

$$\begin{aligned} \overline{A}(\lambda, \mu) &= \text{span}\{v_m^j \mid j, m \in \mathbb{Z}, 0 \leq j \leq p-1\} : \\ L_k v_m^j &= (\lambda + aj + m + \mu k) v_{k+m}^j \quad (0 \leq j \leq p-1), \\ W_k v_m^j &= \frac{1}{a+k} v_{k+m}^{j+1} \quad (0 \leq j < p-1), \quad W_k v_m^{p-1} = \frac{1}{a+k} v_{k+m+pa}^0. \end{aligned} \quad (6.4)$$

Now we give a proof of Theorem 1.2 for $a \in \mathbb{Q} \setminus \mathbb{Z}$. First assume $W_k V^{N_0+N} = 0$ for all $k \in \mathbb{Z}$. Similar to the proof of Theorem 1.2 for the case $a \notin \mathbb{Q}$, we have the result. Now assume $W_{k_0} V^{N_0+N} \neq 0$ for some $k_0 \in \mathbb{Z}$ (by shifting the index, we may assume $N_0 = 0$). This means $N = p-1$ and $W_{k_0} V^{p-1} \subset V^0$. If we re-denote $V^p = V^0$, then we obtain

$$\mu_p = \mu_0. \quad (6.5)$$

Consider all possibilities, we find out that only in the case $\tilde{A}(\lambda, \mu)$, (6.5) can happen, thus we obtain an extra module $\overline{A}(\lambda, \mu)$ defined in (6.4). This completes the proof of the theorem in this case.

Finally we prove Theorem 1.2 for the case $a \in \mathbb{Z}$. In this case we suppose $a = 0$ since $\mathcal{W}(a, b) \simeq \mathcal{W}(0, b)$, and suppose $b \neq 0$ as when $b = 0$, the algebra $\mathcal{W}(0, 0)$ is simply the twisted Heisenberg-Virasoro algebra, whose indecomposable modules of the intermediate series were considered in [7].

The special case $a = 0, b = -1$ has also been solved in [6]. Thus, we assume $b \neq -1$. First we suppose $V = V^0 = \text{span}\{v_m \mid m \in \mathbb{Z}\}$ is a Vir -module of type $A'_{\lambda, \mu}$, and so $L_k v_m = (\lambda + m + \mu k) v_{m+k}$ and $W_k v_m = \mathbf{w}_{k,m} v_{k+m}$. Applying $[L_k, W_0] = bkW_k$, $[W_k, W_0] = 0$ to v_m , comparing the coefficients of v_{m+k} , we have

$$(\lambda + m + \mu k) \mathbf{w}_{0,m} - (\lambda + m + \mu k) \mathbf{w}_{0,m+k} = bk \mathbf{w}_{k,m}, \quad (6.6)$$

$$\mathbf{w}_{k,m} \mathbf{w}_{0,m} = \mathbf{w}_{0,m+k} \mathbf{w}_{k,m}. \quad (6.7)$$

If $\mathbf{w}_{k_0, m_0} \neq 0$ for some k_0, m_0 with $k_0 \neq 0$, then (6.7) gives $\mathbf{w}_{0, m_0} = \mathbf{w}_{0, m_0 + k_0}$, and (6.6) gives $\mathbf{w}_{k_0, m_0} = 0$, which is a contradiction. Thus, $\mathbf{w}_{k, m} = 0$ for all $k \neq 0$. If $b \neq 1$, by applying $[L_1, W_{-1}] = (b-1)W_0$ to v_m , we obtain $\mathbf{w}_{0, m} = (\lambda + m - 1 + \mu)\mathbf{w}_{-1, m} - (\lambda + m + \mu)\mathbf{w}_{-1, m+1} = 0$. Thus, W_k 's act trivially on V . Therefore, we suppose $b = 1$. Then (6.6) gives

$$(\lambda + m + \mu k)(\mathbf{w}_{0, m} - \mathbf{w}_{0, m+k}) = 0, \quad (\lambda + m + k - \mu k)(\mathbf{w}_{0, m} - \mathbf{w}_{0, m+k}) = 0,$$

where the second equation follows from the first by replacing m, k by $m+k, -k$ respectively. Thus

$$(2\lambda + 2m + k)(\mathbf{w}_{0, m} - \mathbf{w}_{0, m+k}) = 0. \quad (6.8)$$

Letting $m = 0$ in (6.8) gives $(2\lambda + k)(\mathbf{w}_{0, 0} - \mathbf{w}_{0, k}) = 0$. If $2\lambda \notin \mathbb{Z}$, then $\mathbf{w}_{0, k} = \mathbf{w}_{0, 0}$ for all $k \in \mathbb{Z}$. If $2\lambda \in \mathbb{Z}$, then $\mathbf{w}_{0, k} = \mathbf{w}_{0, 0}$ for $k \neq -2\lambda$. Letting $m = -2\lambda$ in (6.8) gives $(k - 2\lambda)(\mathbf{w}_{0, -2\lambda} - \mathbf{w}_{0, k-2\lambda}) = 0$ for all $k \in \mathbb{Z}$. Thus $\mathbf{w}_{0, -2\lambda} = \mathbf{w}_{0, k-2\lambda} = \mathbf{w}_{0, 0}$. Hence, we obtain module $\overline{A}(\lambda, \mu, c)$. Now, the proof of Theorem 1.2 is completed.

7 Irreducible modules of the intermediate series

Now we give a proof of Theorem 1.3. First we prove Theorem 1.3(3). Thus suppose $a \notin \mathbb{Q}$. Assume that the set $\{W_k | k \in \mathbb{Z}\}$ acts nontrivially on V , i.e., there exists some nonzero weight vector $v_\lambda \in V_\lambda$ with weight λ such that $W_k v_\lambda \neq 0$ for some k . Note that we have (2.6). Denote $V' = \sum_{j \geq 1, m \in \mathbb{Z}} V_{\lambda + ja + m}$. It is straightforward to verify that $v_\lambda \notin V'$, $0 \neq W_k v_\lambda \in V'$, and V' is a proper $\mathcal{W}(a, b)$ -submodule of V , a contradiction with the irreducibility of V . This proves Theorem 1.3(3).

Theorem 1.3(2) follows immediately from Theorem 1.2. It remains to prove Theorem 1.3(1). Thus assume V is an irreducible Harish-Chandra $\mathcal{W}(a, b)$ -module without highest and lowest weights. We can assume $a \in \mathbb{Q}$ which is written as in (6.1), otherwise the result follows from Theorem 1.3(3) and Mathieu's Theorem ([8], Theorem 1). We claim that for any $m \neq -1, 0$ and $\lambda \in \mathbb{C}$, the linear map

$$\psi_m := L_m|_{V_\lambda} \oplus L_{m+1}|_{V_\lambda} \oplus W_m|_{V_\lambda} \oplus W_{m+1}|_{V_\lambda} : V_\lambda \rightarrow V_{\lambda+m} \oplus V_{\lambda+m+1} \oplus V_{\lambda+a+m} \oplus V_{\lambda+a+m+1} \quad (7.1)$$

is injective. If not, then there exists some $v_0 \in V_\lambda$ such that $L_m v_0 = L_{m+1} v_0 = W_m v_0 = W_{m+1} v_0 = 0$. Without loss of generality, we suppose $m > 0$. Note that when $k \gg 0$, we can always express k as $k = xm + y(m+1)$ for some $x, y \in \mathbb{Z}_+ \setminus \{0\}$, such that L_k, W_k can be generated by $L_m, L_{m+1}, W_m, W_{m+1}$ (note that when $k \gg 0$, we either have $W_k = \frac{1}{a+(1-b)m+bk}[L_{k-m}, W_m]$ with $a + (1-b)m + bk \neq 0$ or $W_k = \frac{1}{a+(1-b)(m+1)+bk}[L_{k-m-1}, W_{m+1}]$ with $a + (1-b)(m+1) + bk \neq 0$). Thus there exists some $K > 0$ such that $L_k v_0 = W_k v_0 = 0$ for all $k > K$. Then as the proof of [12, Proposition 2.1], we obtain a highest weight, a contradiction with the assumption. This proves the claim.

Now fix a weight $\lambda_0 \in P(V)$. We have $P(V) \subset \{\lambda_0 + \frac{i}{p} + m \mid 0 \leq i \leq p-1, m \in \mathbb{Z}\}$ by (2.6). Denote $N = \sum_{i=0}^{p-1} \sum_{j=-1}^1 \dim V_{\lambda_0 + \frac{i}{p} + j} < \infty$. Then for any $\lambda = \lambda_0 + \frac{i}{p} + m \in P(V)$, we always have $\dim V_\lambda \leq N$ (which is obvious if $m = 0, -1$, and which follows from the injectivity of the map ψ_{-m} in (7.1) if $m \neq 0, 1$). Thus V is uniformly bounded, and the proof of Theorem 1.3 is completed.

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